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Strong interaction of solitons governed by Korteweg–de Vries equation

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Abstract. Properties of the strong nonlinear interaction between solitons have been investigated in detail. In this case, the solitons behave exactly like particles colliding elastically in a potential field; they never pass through each other, but bounce back at a distance. Momentum transfer is considered to be the basic mechanism of interaction. Thus, the velocity of propagation of each soliton is a function of time during collision. Explicit expressions for the minimum distance between solitons, the time when this distance occurs, and the amplitudes of solitons at this time have all been derived. Reversibility and other important properties have also been proved.

1. Introduction

During the past two decades there has been a great deal of interest in the study of nonlinear dispersive waves. Many authors have adopted the Korteweg–de Vries equation as a model for such physical systems. This subject is well known. Work up to 1971 has been thoroughly reviewed by Jeffrey and Kakutani (1972).

Zabusky and Kruskal (1965) made an extensive numerical study of the problem and discovered many interesting features. One of the most remarkable characteristics is that the Korteweg–de Vries equation has a stable pulse-like solution known as a soliton. Analytically, Lax (1968) proved the Zabusky and Kruskal observation for a double-soliton system.

More recently, Hirota (1971) obtained an exact solution for N solitons. Later, Wadati and Toda (1972) proved the same solution based on the scheme proposed by Gardner *et al* (1967).

Gibbon and Eilbeck (1972) found that the asymptotic effect on phase shifts is linear for nonlinear interaction. Wadati and Toda (1972) further proved that the algebraic sum of the phase shifts of all solitons is conserved which indicates the uniform motion of the centre of the system.

In the present paper, we investigate the properties when strong nonlinear interaction between solitons occurs. In this case, solitons behave exactly like particles colliding elastically in a potential field; they never pass through each other, but bounce back at a distance. Explicit expressions for this minimum distance, the time when this distance occurs, and the amplitudes of solitons at this time are all derived. Other important properties are also proved analytically.

2. Preliminary remarks

Consider a system of N solitons governed by the Korteweg–de Vries equation of the form

$$u_t + uu_x + \beta u_{xxx} = 0, \tag{2.1}$$

subject to the boundary conditions that $u(x, t)$ vanishes exponentially at $x = \pm\infty$. An exact solution, through the transformation (Hirota 1971, Wadati and Toda 1972)

$$u(x, t) = 12\beta(\ln f)_{xx}, \tag{2.2}$$

may be expressed as

$$f(x, t) = 1 + \sum_{r=1}^{N} \sum_{{}_N C_r} \eta^2(i_1 \ldots i_r) \exp\left(\sum_{i=i_1}^{i_r} \kappa_i(x - C_i t - a_i) \right), \tag{2.3}$$

$$\eta(i_1 \ldots i_r) = \prod_{j<k}^{(r)} \eta_{jk} = \prod_{j<k}^{(r)} \frac{\kappa_j - \kappa_k}{\kappa_j + \kappa_k}, \tag{2.3a}$$

$$\kappa_i = (C_i/\beta)^{1/2}, \tag{2.3b}$$

where ${}_N C_r$ indicates summation over all possible combinations of r elements (designated as $i_1, i_2, \ldots, i_r$) taken from N, and (r) indicates the product of all possible pairs out of these r elements; C_i is the initial propagation velocity of ith soliton at $t = -\infty$, and a_i a constant related to its position. It is understood that for $r = 1$, $\eta = 1$.

Without nonlinear interaction, the function f will be

$$f_0 = \prod_{i}^{N} \{1 + \exp[\kappa_i(x - C_i t - a_i)]\}.$$

Thus, $1 - \eta^2(i_1 \ldots i_r)$ may be interpreted as an interaction parameter for r solitons. When solitons have comparable initial amplitudes, the interaction is strong, otherwise it is weak.

Numerical studies on the interaction of two solitons have shown that when the larger soliton approaches the smaller one from the left, it starts to shrink, while the smaller soliton starts to grow. In cases when one soliton has much larger amplitude than the other (weak interaction), the larger soliton swallows up the smaller one during the collision, and re-emits it later. In cases when the two solitons are of comparable amplitudes (strong interaction), they interact at a distance and separate. As an illustrative example, positions of two interacting solitons relative to the centre of the system are shown in figure 1. It indicates that the phenomenon of strong interaction is very similar to the elastic collision of two particles in a potential field.

Hence, we state the basic postulate as follows:

The mechanism of interaction between solitons is based on the momentum transfer. Solitons in collision have a tendency to repel each other. As a consequence, one soliton slows down while the other speeds up. Thus, the velocity of propagation of each soliton is a function of time, $c_i(t)$, subject to the boundary conditions (t as independent variable):

$$\text{At } t = -\infty, \qquad c_i = C_i;$$
$$\text{At } t = \infty, \qquad c_i = C_{N+1-i}.$$

When the initial velocity ratio of solitons is below a certain limit, the two solitons become identical at a minimum distance, and bounce off from each other. If the initial

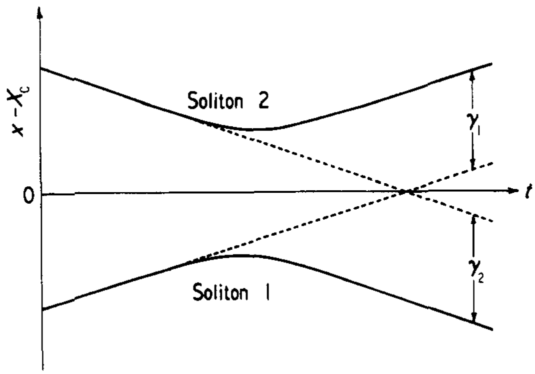

Figure 1. Typical trajectories of solitons in strong interaction.

velocity ratio is above this limit, the larger soliton is impulsive enough to overcome the repulsion and passes through the other. As an alternative interpretation, one may also assume that solitons always pass through each other.

3. Properties at the centre of the system

The centre of the soliton system, which moves at uniform velocity, is defined as (Wadati and Toda 1972)

$$X_C = \frac{\Sigma_i^N \kappa_i(C_i t + \chi_i)}{\Sigma_i^N \kappa_i}, \tag{3.1}$$

where χ_i, defined as

$$\chi_i = a_i - \frac{2}{\kappa_i} \sum_{j<i}^{N} \ln \eta_{ij}, \tag{3.2}$$

is the position at $t = 0$ on the asymptote to the trajectory of the ith soliton.

We number a train of solitons from left to right in order, and without loss of generality we assume that the soliton on the left always has a larger amplitude than the one on the right. For simplicity of mathematical expression, we introduce

$$\xi_i = \kappa_i(x - C_i t - a_i) \tag{3.3a}$$

and

$$E_i = \exp \xi_i. \tag{3.3b}$$

At $x = X_C$, substituting expression (3.1) into equation (3.3a), we obtain

$$\xi_j = \frac{\Sigma_i^N \kappa_i \kappa_j[(C_i - C_j)t + \chi_i - \chi_j]}{\Sigma_i^N \kappa_i} - 2 \sum_{i<j}^{N} \ln \eta_{ij}. \tag{3.4}$$

As we sum j over all solitons, the first part in equation (3.4) vanishes; thus we have

$$\sum_{j}^{N} \xi_j = -2 \sum_{i<j}^{(N)} \ln \eta_{ij}.$$

This proves that at the centre of the system ($x = X_C$), the following condition is satisfied for all time:

$$\left(\prod_{i<j}^{(N)} \eta_{ij}^2\right)\left(\prod_{j}^{N} E_j\right) = 1. \tag{3.5}$$

Since the interaction of a soliton pair is the basic structure of the interaction of the entire system, we shall deal mainly with the two-soliton system.

For a double-soliton system, by equations (2.2), (2.3), and (3.1), we write

$$u(x, t) = 12\beta[\kappa_1^2 E_1(1+\eta^2 E_2^2)+\kappa_2^2 E_2(1+\eta^2 E_1^2)+2(\kappa_1-\kappa_2)^2 E_1 E_2] \times (1+E_1+E_2+\eta^2 E_1 E_2)^{-2}, \tag{3.6}$$

$$u_x(x, t) = 12\beta\{\kappa_1^3 E_1[1-E_1+\eta^2 E_2^3(1-\eta^2 E_1)]+\kappa_2^3 E_2[1-E_2+\eta^2 E_1^3(1-\eta^2 E_2)] + (\kappa_1+\kappa_2)(3\kappa_1^2-7\kappa_1\kappa_2+3\kappa_2^2)E_1E_2(1-\eta^2 E_1 E_2) - \eta^2(\kappa_1-\kappa_2)(3\kappa_1^2+7\kappa_1\kappa_2+3\kappa_2^2)E_1E_2(E_1-E_2)\} \times (1+E_1+E_2+\eta^2 E_1 E_2)^{-3}, \tag{3.7}$$

$$X_C = [C_1+C_2-(C_1C_2)^{1/2}]t+[C_1a_1-C_2a_2+(C_1C_2)^{1/2}(a_2-a_1)](C_1-C_2)^{-1} - \frac{2(\ln\eta)}{\kappa_1+\kappa_2}. \tag{3.8}$$

Condition (3.5) becomes simply

$$\eta^2 E_1 E_2 = 1. \tag{3.9}$$

Under condition (3.9), equations (3.6) and (3.7) may be reduced to

$$u(X_C, t) = 12\beta[(\kappa_1^2+\kappa_2^2)(E_1+E_2)+2(\kappa_1+\kappa_2)^2](E_1+E_2+2)^{-2}, \tag{3.10}$$

$$u_x(X_C, t) = 12\beta(E_2-E_1)[(\kappa_1^3-\kappa_2^3)(E_1+E_2)+2(\kappa_1-\kappa_2)(\kappa_1^2+3\kappa_1\kappa_2+\kappa_2^2)] \times (E_1+E_2+2)^{-3}. \tag{3.11}$$

It is apparent from equation (3.11) that the minimum of u occurs at X_C when and only when E_1 and E_2 are equal. At this moment (say $t = T_*$), by condition (3.9) we have

$$E_1 = E_2 = 1/\eta. \tag{3.12}$$

By equations (3.3), (3.8), and (3.12), we obtain

$$\xi_1(X_C, T_*) = (\kappa_1-\kappa_2)(C_1C_2)^{1/2}\left(\frac{a_2-a_1}{C_1-C_2} - T_*\right) - \frac{2\kappa_1}{\kappa_1+\kappa_2}\ln\eta = -\ln\eta,$$

which yields

$$T_* = \left[a_2-a_1-\left(\frac{1}{\kappa_2}-\frac{1}{\kappa_1}\right)\ln\eta\right](C_1-C_2)^{-1}. \tag{3.13}$$

At this time ($t = T_*$), by equations (3.10) and (3.9) we find that

$$u_{\min}(T_*) = 3(C_1 - C_2), \tag{3.14}$$

which occurs at

$$X_{\mathrm{m}} = X_{\mathrm{C}}(T_*) = \frac{C_1 a_2 - C_2 a_1}{C_1 - C_2} - \frac{1 + \kappa_1/\kappa_2 + \kappa_2/\kappa_1}{\kappa_1 + \kappa_2} \ln \eta. \tag{3.15}$$

4. Solitons at minimum distance

The distance between two solitons is defined as

$$L(t) = X_2(t) - X_1(t),$$

where $X_i(t)$ denotes the position of the ith soliton. Here and throughout this paper, we consider the points where u is a maximum as soliton positions, since we have not yet found any other better way to define them. Differentiating the above expression, we have

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}X_2}{\mathrm{d}t} - \frac{\mathrm{d}X_1}{\mathrm{d}t} = c_2(t) - c_1(t).$$

Thus, the minimum of L occurs when the velocities of propagation c_1 and c_2 are equal (both equal to the velocity of the centre of system). Physically speaking, in the beginning when $c_1 > c_2$ the solitons become closer and closer; after a certain time due to momentum transfer so that $c_1 < c_2$, the solitons become more and more distant from each other.

We are particularly interested in the situation when solitons are at the closest position, since at that moment the interaction reaches its peak. The time T_* as given by expression (3.13), which will be proved in § 6, is the moment when two solitons are at minimum distance.

Define

$$x^* = x - X_{\mathrm{C}}(t)$$

and

$$X_i^*(t) = X_i(t) - X_{\mathrm{C}}(t).$$

At $t = T_*$, by equations (3.3a), (3.13), and (3.15), we get

$$\xi_1(x, T_*) = \kappa_1 x^* - \ln \eta, \tag{4.1a}$$

$$\xi_2(x, T_*) = \kappa_2 x^* - \ln \eta. \tag{4.1b}$$

Thus, we have

$$\frac{\xi_1}{\kappa_1} - \frac{\xi_2}{\kappa_2} = \left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right) \ln \eta,$$

or, by expression (3.3b),

$$(\eta E_1)^{1/\kappa_1} = (\eta E_2)^{1/\kappa_2} \tag{4.2}$$

for all x.

For mathematical simplicity, we introduce two variables

$$z(x^*, T_*) = \eta E_2(x^*, T_*) \tag{4.3a}$$

and

$$p = \kappa_1/\kappa_2. \tag{4.3b}$$

Then, by equation (4.2), equations (3.6) and (3.7) become

$$u(z) = 12C_2\eta[z(z^p-1)^2+p^2z^p(z+1)^2](z+z^p+\eta z^{p+1}+\eta)^{-2}, \tag{4.4}$$

$$\begin{aligned} u_x(z) = 12C_2\eta\kappa_2 z\{&\eta - z + z^{3p}(1-\eta z) + p^3z^{p-1}[\eta+z^3-z^p(1+\eta z^3)] \\ &- (p-1)(3p^2+7p+3)\eta z^{p+1}(z^{p-1}-1)-(p+1)(3p^2-7p+3)z^p(z^{p+1}-1)\} \\ &\times (z+z^p+\eta z^{p+1}+\eta)^{-3}. \end{aligned} \tag{4.5}$$

By equation (4.1*b*), we find that

$$z(x^*)z(-x^*) = 1. \tag{4.6}$$

By equation (4.4), we verify that indeed

$$u(1/z) = u(z),$$

which implies that

$$u(-x^*) = u(x^*).$$

This proves that at $t = T_*$, the wave profile is symmetrical to the centre of the system.

Thus, we have

$$A_1 = A_2 \qquad \text{and} \qquad X_1^* = -X_2^*. \tag{4.7}$$

In order to investigate the distance between two solitons, we have to find the solution for

$$u_x(z) = 0,$$

where the expression for $u_x(z)$ is given by equation (4.5). The trivial solutions of $z = \infty$ and 0 represent the minima at $x = \pm\infty$. The solution $z = 1$, as proved in § 3, represents the minimum at $x = X_C(T_*)$. There are two other solutions, denoted by $Z = z(X_i)$, which represent the maxima. Following equation (4.5), the solution Z must satisfy

$$\begin{aligned} Z^{3p}-Z-\eta(Z^{3p+1}-1)+p^3Z^{p-1}(Z^3-Z^p-\eta Z^{3+p}+\eta)-(p-1)(3p^2+7p+3)\eta Z^p(Z^p-Z) \\ -(p+1)(3p^2-7p+3)Z^p(Z^{p+1}-1) = 0. \end{aligned} \tag{4.8}$$

The minimum distance between two solitons is, by definition and equation (4.7),

$$L_{\min} = X_2(T_*)-X_1(T_*) = 2X_2^*(T_*).$$

By equation (4.1*b*), we get

$$\kappa_2 L_{\min} = 2 \ln Z(X_2). \tag{4.9}$$

Since $\eta = (p-1)/(p+1)$, equation (4.8) shows that Z is a function of p only. Therefore, the parameters of locations ($\kappa_2 X_i$) and amplitudes ($A/3C_2$) of solitons, and their distance ($\kappa_2 L_{\min}$) are all functions of the initial velocity ratio (C_1/C_2) only. (The word initial here refers to $t = -\infty$.)

Here, we shall discuss only the solution $Z \geqslant 1$, which represents the location of soliton 2, since by equations (4.6) and (4.7) the other solution $Z(X_1)$ is just the reciprocal of this solution $Z(X_2)$.

When the velocity ratio C_1/C_2 approaches unity as its limit, $Z(X_2)$ approaches infinity (figure 2). On the other hand, when $C_1/C_2 = 3$, we have $Z = 1$. No solution exists for $C_1/C_2 > 3$.

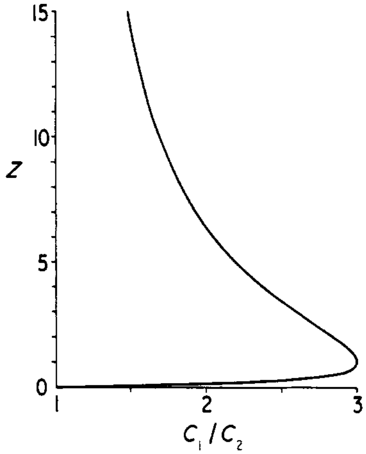

Figure 2. Solution of $Z(X_2, T_*)$ as function of velocity ratio.

It must be noted that imaginary roots of Z for $C_1/C_2 > 3$ do not imply non-existence of two-soliton solutions. In arriving at equation (4.8), the condition of profile symmetry at $t = T_*$ has been taken into consideration. It indicates that as $C_1 > 3C_2$, the weak interaction occurs. In this case when two solitons coincide with each other, one soliton still possesses larger momentum than the other, thus the situation of equal amplitudes can never be reached.

For $\eta \ll 1$ (ie, $C_1 \simeq C_2$), it is shown in the appendix that an approximate solution can be found:

$$Z(X_2) = \frac{1+(\frac{1}{2}\eta)^{2\eta}}{\eta}. \tag{4.10}$$

By equation (4.9), we have

$$\kappa_2 L_{\min} = -2 \ln \eta. \tag{4.11}$$

It is also shown in the appendix that

$$A/3C_2 = 1+2\eta. \tag{4.12}$$

5. Limiting case of strong interaction

It has been mentioned previously that due to the interaction between two solitons (note that the amplitude of the left soliton is initially greater than the right one), the left soliton reduces its velocity of propagation while the right one increases its velocity. If the difference of initial velocities is not very large, then the velocities at some time later become equal before the left soliton can overtake the other. Thereafter, since the velocity of left soliton becomes smaller than the right one, they start to separate from each other. This is the case of strong interaction. Otherwise, it is the case of weak interaction. Therefore, $C_1 = 3C_2$, for which the solitons coincide when their velocities become equal, is the limiting case of strong interaction. Mathematically, by equation (4.8) we obtain

$$Z(T_*) = 1.$$

By equation (4.9),

$$L_{\min} = 0.$$

By equation (4.4),

$$A/3C_2 = 2.$$

This is probably the best case to study the nonlinear interaction when one soliton coinicdes with the other. The wave profile may be found as follows.

By expression (4.3*a*),

$$z = \eta \exp[\kappa_2(x - C_2 T_* - a_2)] = Z \exp[\kappa_2(x - X_2)].$$

By equation (4.2),

$$z^p = \eta \exp[\kappa_1(x - C_1 T_* - a_1)] = Z^p \exp[\kappa_1(x - X_1)].$$

Here,

$$X_1 = X_2 = X_C(T_*).$$

Due to the fact that $Z = 1$ for this case, we have

$$z = \exp(\kappa_2 x^*),$$

$$z^p = \exp(\kappa_1 x^*).$$

Thus, substituting these expressions into equation (4.4) and expressing in terms of hyperbolic functions, we obtain

$$\frac{u}{A} = \frac{3\,\mathrm{sech}^2(\frac{1}{2}\kappa_1 x^*) + \tanh^2(\frac{1}{2}\kappa_1 x^*)\,\mathrm{sech}^2(\frac{1}{2}\kappa_2 x^*)}{[\sqrt{3} - \tanh(\frac{1}{2}\kappa_1 x^*)\tanh(\frac{1}{2}\kappa_2 x^*)]^2}. \tag{5.1}$$

The wave profile has a very flat top (figure 3). It is interesting to note that zero curvature occurs at the centre; ie,

$$u_{xx} = 0 \qquad \text{at} \qquad x^* = 0.$$

This may be interpreted as the phenomenon that two maxima and one minimum fall on a single point.

6. Reversibility and asymptotic behaviour

Now we shall prove that the solution $u(x, t)$ satisfies the condition of reversibility

$$u(X_m - \theta, T_* - \tau) = u(X_m + \theta, T_* + \tau), \tag{6.1}$$

where the variables θ and τ are both positive, and X_m and T_* are constant as given by equations (3.15) and (3.13).

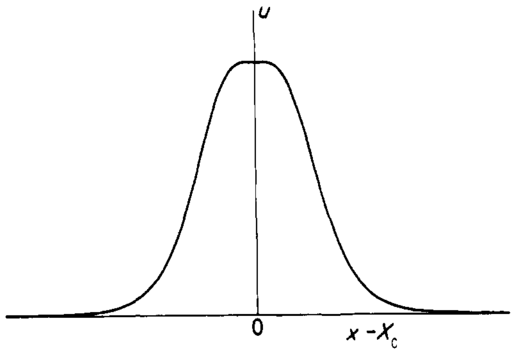

Figure 3. Wave profile for $C_1/C_2 = 3$ when solitons coincide.

Due to symmetry of $u(x^*, T_*)$ as proved in § 4, condition (6.1) is satisfied for $\tau = 0$. If $u(X_m + \theta, T_* + \tau)$ is a solution of equation (2.1), then $u(X_m - \theta, T_* - \tau)$ is also a solution of equation (2.1). Since condition (6.1) holds for $\tau = 0$, it must also be satisfied for $\tau \neq 0$.

As a corollary, we may state that

$$L(T_* - \tau) = L(T_* + \tau).$$

Thus, at $t = T_*$,

$$\frac{dL}{dt} = 0.$$

That is to say, at this time L is either a maximum or a minimum. From the physical point of view, the left soliton moves faster than the right one when $t < T_*$. Therefore, L must be the minimum at $t = T_*$. This proves the statement which we made in § 4.

In view of reversibility, we only have to study the wave motion from $t = -\infty$ up to T_*. Consider the invariant for either soliton

$$\int_{-\infty}^{\infty} c_i(t)\, dt = \int_{-\infty}^{0} [c_1(\tau) + c_2(\tau)]\, d\tau = X_i(\infty) - X_i(-\infty), \tag{6.2}$$

where the subscript i may be either 1 or 2. Asymptotically for $\tau_\infty \to \infty$, we have

$$X_1(-\infty) = C_1(T_* - \tau_\infty) + a_1,$$
$$X_1(\infty) = C_2(T_* + \tau_\infty) + a_2.$$

Thus, by (6.2) and (3.13), we obtain

$$\int_{-\infty}^{0} [c_1(\tau) + c_2(\tau)]\, d\tau = (C_1 + C_2)\tau_\infty + \left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right) \ln \eta. \tag{6.3}$$

It indicates in equation (6.3) that for $\tau < \infty$,

$$c_1(t) + c_2(t) < C_1 + C_2.$$

The term

$$\left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right) \ln \eta,$$

an invariant for either soliton, is the actual phase shift if the soliton is considered in uniform motion ($c_i(t) = C_i$ when $t < T_*$ and $c_i(t) = C_{3-i}$ when $t > T_*$).

Appendix. Approximate solution of $Z(X_2, T_*)$ for $\eta \ll 1$

When the amplitudes of two solitons are almost equal, η is very small and $Z(X_2, T_*)$ is very large. Assume

$$Z(X_2, T_*) = \phi(\eta)/\eta, \tag{A.1}$$

where $\phi(\eta)$ is of the order of unity, which may be verified by numerical computations of a few points. The parameter p may be approximated as

$$p = \frac{1+\eta}{1-\eta} = 1+2\eta. \tag{A.2}$$

Substituting expressions (A.1) and (A.2) into equation (4.8), and neglecting smaller quantities of higher orders, we obtain

$$(1-\phi)Z^{3p}+(1-\phi Z^{p-1})Z^{p+2}+2Z^{2p+1} = 0. \tag{A.3}$$

Re-arranging equation (A.3) and dividing by $Z^{p+2}(Z^{p-1}+1)$, we have

$$Z^{p-1}+1 = \phi Z^{p-1},$$

or

$$(\phi-1)Z^{p-1} = 1. \tag{A.4}$$

Multiplying equation (A.4) by $\eta^{2\eta}$, we get

$$(\phi-1)\phi^{2\eta} = \eta^{2\eta}. \tag{A.5}$$

Since

$$\lim_{\eta\to 0} \eta^{2\eta} = 1,$$

by equation (A.5) we have

$$\lim_{\eta\to 0} \phi(\eta) = 2.$$

We assume $\phi(\eta) = 2-\alpha(\eta)$, where $\alpha(\eta) \ll 1$. By equation (A.5), we obtain approximately

$$\alpha(\eta) = 1-(\tfrac{1}{2}\eta)^2,$$

or

$$\phi(\eta) = 1+(\tfrac{1}{2}\eta)^2. \tag{A.6}$$

This leads to equation (4.10).

Similarly, by equation (4.4) we have

$$\frac{A}{3C_2} = \frac{4\phi Z^{2\eta}(Z^{2\eta}+p^2)}{[(1+\phi)Z^{2\eta}+1]^2}. \tag{A.7}$$

Since $(\frac{1}{2}\eta)^{2\eta} \simeq 1$, we may approximate it by

$$(\tfrac{1}{2}\eta)^{2\eta} = 1+2\eta\ln(\tfrac{1}{2}\eta).$$

Thus, by equation (4.10), $Z(X_2, T_*)$ may be expressed as

$$Z = \frac{2}{\eta}[1+\eta \ln(\tfrac{1}{2}\eta)].$$

Note that

$$\lim_{\eta \to 0} \eta \ln \eta = 0.$$

Neglecting small quantities of second and higher orders, we get

$$Z^{2\eta} = (2/\eta)^{2\eta}, \tag{A.8}$$

$$p^2 = 1+4\eta. \tag{A.9}$$

Substituting expressions (A.6), (A.8), and (A.9) into equation (A.7), we prove equation (4.12).

References

Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095–7
Gibbon J D and Eilbeck J C 1972 *J. Phys.* A: *Gen. Phys.* **5** L132–5
Hirota R 1971 *Phys. Rev. Lett.* **27** 1192–4
Jeffrey A and Kakutani T 1972 *SIAM Rev.* **14** 582–643
Lax P D 1968 *Commun. Pure Appl. Math.* **21** 467–90
Wadati M and Toda M 1972 *J. Phys. Soc. Japan* **32** 1403–11
Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240–3